Math 2A – Chapter 9 Test – Fall ’09 Name________________________________

Directions:

Write all responses on separate paper.

Show all work for credit.

Do not use a calculator.

1. Consider the vectors $\vec{u} = \langle 0,2,5 \rangle$ and $\vec{v} = \langle 0,-2,3 \rangle$.
   a. Sketch a graph of these two vectors, together with the vectors $\vec{u}+\vec{v} = \overrightarrow{u+v}$ and $\vec{u}-\vec{v} = \overrightarrow{u-v}$ as the sides and diagonals of a parallelogram.
   b. Find $\left|\overrightarrow{u+v}\right|$ and $\left|\overrightarrow{u-v}\right|$ and use these to verify the parallelogram law, $\left|\overrightarrow{u+v}\right|^2 + \left|\overrightarrow{u-v}\right|^2 = 2\left|\vec{u}\right|^2 + 2\left|\vec{v}\right|^2$

2. Consider the points $P(1,2,3)$, $Q(5,1,0)$ and $R(2,4,0)$
   a. Compute the terminal point of the vector for each of $\overrightarrow{RP}$, $\overrightarrow{RQ}$ and $\overrightarrow{QP}$, assuming the initial point is at (0,0,0). Hint: this is the brocket <> form of the vector.
   b. Write a parametric equation(s) for the line through P and Q.
   c. Find an equation for the plane containing P, Q and R.

3. A force $\vec{F}$ of magnitude $\sqrt{3}$ Newtons is applied in the direction from (0,0,0) to (1,1,1) and moves a mass of 5kg from (0,0,0) to (1,2,3).
   a. What is the radian measure of the angle between $\vec{F}$ and the displacement vector? Use inverse trig notation and simplified radicals, if necessary, to write your answer.
   b. What work is done in this case? Assume all coordinates are in meters and give your answer in Joules.

4. Consider the lines $\vec{r}_1(t) = \langle 0,1,3 \rangle + \langle 0,2,1 \rangle t$ and $\vec{r}_2(t) = \langle 1,0,2 \rangle + \langle 3,0,1 \rangle t$
   a. Find a vector $\vec{u}$ perpendicular to both $\vec{r}_1(t)$ and $\vec{r}_2(t)$.
   b. Find the length of the projection of a vector from $\vec{r}_1(0)$ to $\vec{r}_2(0)$ onto $\vec{u}$.

      What is the significance of this length in terms of the lines $\vec{r}_1(t)$ and $\vec{r}_2(t)$?

5. Find the coordinates of the *x*-intercept, *y*-intercept and *z*-intercept of the plane $x+2y+3z=6$.
   Draw the points in a rendering of three space and the triangle with those vertices.

6. Find and sketch the domain of the function $f(x,y) = \sqrt{\sin\left[\pi\left(x^2+y^2\right)\right]}$

7. Consider the surface described by $2x^2+y^2-2y=z^2$
   a. Describe cross sections parallel to the *xy*-plane. What shape are they? Where are they centered?
   b. Describe cross sections parallel to the *xz*-plane. What shape are they? Where are they centered?

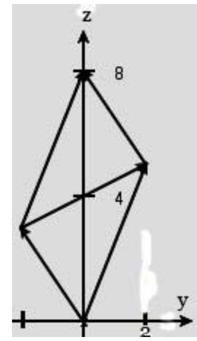
8. Consider the function given in spherical coordinates: $\rho = \dfrac{4\sin\theta}{\sin\phi}$.

   Write the equation in rectangular form. Describe the shape of the surface.

Math 2A – Chapter 9 Test Solutions – Fall '09

1. Consider the vectors $\vec{u}=\langle 0,2,5\rangle$ and $\vec{v}=\langle 0,-2,3\rangle$.

   a. Sketch a graph of these two vectors, together with the vectors $\vec{u}+\vec{v}=\overrightarrow{u+v}$ and $\vec{u}-\vec{v}=\overrightarrow{u-v}$ as the sides and diagonals of a parallelogram.
   SOLN: As shown in the diagram at right, these vectors are in the yz-plane with the sum $\vec{u}+\vec{v}=\langle 0,0,8\rangle$ parallel to the *z*-axis and $\vec{u}-\vec{v}=\langle 0,4,2\rangle$



   b. Find $\left|\overrightarrow{u+v}\right|$ and $\left|\overrightarrow{u-v}\right|$ and use these to verify the parallelogram law

   SOLN: $\left|\overrightarrow{u+v}\right|^2+\left|\overrightarrow{u-v}\right|^2=2\left|\vec{u}\right|^2+2\left|\vec{v}\right|^2$

$$\left|\overrightarrow{u+v}\right|^2+\left|\overrightarrow{u-v}\right|^2 = \ 0^2+0^2+8^2\ +\ 0^2+4^2+2^2$$
$$=64+20=58+26$$
$$=2\ \ 0^2+5^2+2^2\ \ +2\ \ 0^2+\ -2\ ^2+3^2$$
$$=2\left|\vec{u}\right|^2+2\left|\vec{v}\right|^2$$

2. Consider the points $P(1,2,3)$, $Q(5,1,0)$ and $R(2,4,0)$

   a. Compute the terminal point of the vector for each of $\overrightarrow{RP}$, $\overrightarrow{RQ}$ and $\overrightarrow{QP}$, assuming the initial point is at (0,0,0). Hint: this is the brocket <> form of the vector.
   SOLN: $\overrightarrow{RP}=\langle 1,2,3\rangle-\langle 2,4,0\rangle=\langle -1,-2,3\rangle$,
   $\overrightarrow{RQ}=\langle 5,1,0\rangle-\langle 2,4,0\rangle=\langle 3,-3,0\rangle=3\langle 1,-1,0\rangle$ and
   $\overrightarrow{QP}=\langle 5,1,0\rangle-\langle 1,2,3\rangle=\langle 4,-1,-3\rangle$

   b. Write a parametric equation(s) for the line through P and Q.
   SOLN: $\vec{r}\ \ t\ \ =\langle 1,2,3\rangle+\langle 4,-1,-3\rangle t$ will do.

   c. Find an equation for the plane containing P, Q and R.
   SOLN: $\vec{n}=\langle 1,-1,0\rangle\times\langle 1,2,-3\rangle=\langle 3,3,3\rangle$ and so $x+y+z-6=0$ fits the points.

3. A force $\vec{F}$ of magnitude $\sqrt{3}$ Newtons is applied in the direction from (0,0,0) to (1,1,1) and moves a mass of 5kg from (0,0,0) to (1,2,3).

   a. What is the radian measure of the angle between $\vec{F}$ and the displacement vector?
   Use inverse trig notation and simplified radicals, if necessary, to write your answer.
   SOLN: This is sneaky, because the 5 kg doesn't matter at all. Work =

$$\vec{F}\bullet\vec{D}=\left|\vec{F}\right|\left|\vec{D}\right|\cos\theta=\sqrt{3}\sqrt{14}\cos\theta=\langle 1,1,1\rangle\bullet\langle 1,2,3\rangle=6\Leftrightarrow\theta=\cos^{-1}\left(\frac{6}{\sqrt{42}}\right)=\cos^{-1}\left(\frac{\sqrt{42}}{7}\right)$$

   b. What work is done in this case? Assume all coordinates are in meters and give your answer in Joules.
   SOLN: 6 Joules.

4. Consider the lines $\vec{r}_1(t)=\langle 0,1,3\rangle+\langle 0,2,1\rangle t$ and $\vec{r}_2(t)=\langle 1,0,2\rangle+\langle 3,0,1\rangle t$

   a. Find a vector $\vec{u}$ perpendicular to both $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

   SOLN: $\vec{u}=\langle 0,2,1\rangle\times\langle 3,0,1\rangle=\langle 2,3,-6\rangle$ will do.

   b. Find the length of the projection of a vector from $\vec{r}_1(0)$ to $\vec{r}_2(0)$ onto $\vec{u}$.

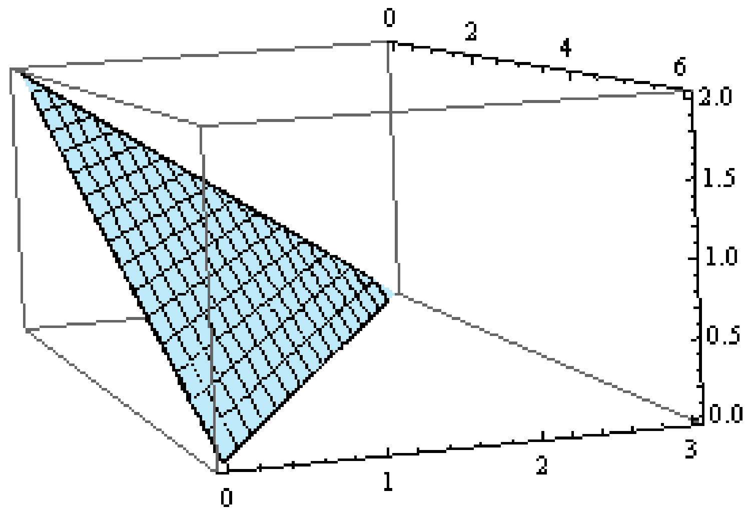
   What is the significance of this length in terms of the lines $\vec{r}_1(t)$ and $\vec{r}_2(t)$?

   SOLN: The vector from $\vec{r}_1(0)$ to $\vec{r}_2(0)$ is $\langle 1,0,2\rangle-\langle 0,1,3\rangle=\langle 1,-1,-1\rangle$ so the length of the projection onto $\vec{u}$ is $\left|\dfrac{\langle 1,-1,-1\rangle\bullet\langle 2,3,-6\rangle}{|\langle 2,3,-6\rangle|}\right|=\left|\dfrac{2-3+6}{7}\right|=\dfrac{5}{7}$ is the shortest distance between the lines.

5. Find the coordinates of the *x*-intercept, *y*-intercept and *z*-intercept of the plane $x+2y+3z=6$. Draw the points in a rendering of three space and the triangle with those vertices.

   SOLN: $(6,0,0)$, $(0,3,0)$, $(0,0,2)$ are the intercepts. Here's a Mathematica rendering:

   ParametricPlot3D[{{6-6u,0,2u},{0,3u,2-2u},{6-6u,3u,0},{6-6u-3v,3u,v}},{u,0,1},{v,0,2-2u}]



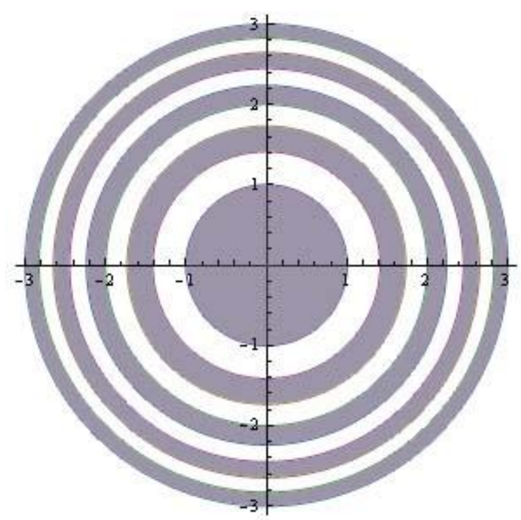
6. Find and sketch the domain of the function

   $f(x,y)=\sqrt{\sin\left[\pi\left(x^2+y^2\right)\right]}$

   SOLN: For the function to be real valued we need

   $\sin\left[\pi\left(x^2+y^2\right)\right]\ge 0\Leftrightarrow 2k\le x^2+y^2\le 2k+1$

   That is, that the square of the distance to the origin of a point in the *xy*-plane is bigger than some even number and less than or equal to the subsequent odd number. $\sqrt{2k}\le D\le\sqrt{2k+1}$. This leads to a sequence of concentric annuli of radii increasing at a decreasing rate. The shaded regions in the diagram at right illustrate the domain – the pattern continues as the distance from the origin goes to infinity:



7. Consider the surface described by $2x^2 + y^2 - 2y = z^2$

   a. Describe cross sections parallel to the *xy*-plane. What shape are they? Where are they centered?
   SOLN: Planes parallel to the *xy*-plane have a constant *z* value. Substituting, *z* = *c* in the equation, we get $2x^2 + y^2 - 2y = c^2 \Leftrightarrow 2x^2 + (y-1)^2 = c^2 + 1$. This is the equation for an ellipse centered at (0,1,*c*).

   b. Describe cross sections parallel to the *xz*-plane. What shape are they? Where are they centered?
   SOLN: Planes parallel to the *xz*-plane have a constant *y* value. Substituting, *y* = *c* in the equation, we get $2x^2 + c^2 - 2c = z^2 \Leftrightarrow 2x^2 - z^2 = 2c - c^2$. These are hyperbolas centered at (0,*c*,0).

   All together, the surface is a hyperboloid of one sheet. You can see the hyperbolas in the wireframe of the Mathematica plot below. The cross section with *z* = 1 is also shown.

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ParametricPlot3D[{{u, v + 1, Sqrt[2 * u^2 + v^2 − 1]}, {u, v + 1, −Sqrt[2 * u^2 + v^2 − 1]}, {Cos[Pi * u], 1
        + Sqrt[2] * Sin[Pi * u], 1}}, {u, −2,2}, {v, −2,2}, PlotStyle
        → {Thickness[0.01], Thickness[0.01], Thickness[0.04]}]
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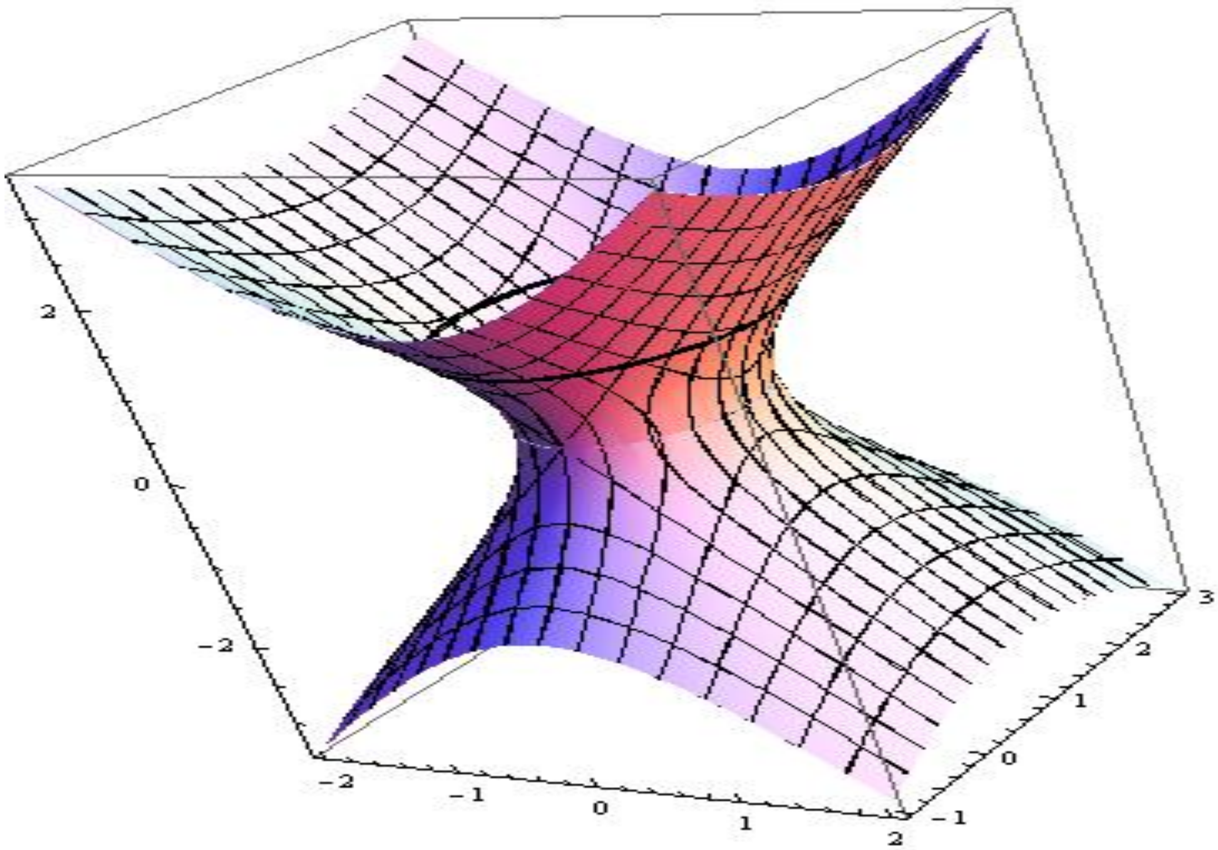

8. Consider the function given in spherical coordinates: $\rho = \dfrac{4\sin\theta}{\sin\phi}$.

   Write the equation in rectangular form. Describe the shape of the surface.

   SOLN: $\rho = \dfrac{4\sin\theta}{\sin\phi} \Leftrightarrow \rho\sin\phi = 4\sin\theta \Leftrightarrow r = 4\dfrac{y}{r} \Leftrightarrow r^2 = 4y \Leftrightarrow x^2 + y^2 = 4 \Leftrightarrow \boxed{x^2 + (y-2)^2 = 4}$

   The last equation describes a circular cylinder of radius 2 perpendicular to the *xy*-plane with the axis along *y* = 2 in the *xz*-plane.